Feedback models and stability analysis of three economic paradigms

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Abstract — In this paper, simple mathematical models from Control Theory are applied to three very important economic paradigms, namely (a) minimum wages in self-regulating markets, (b) market-versus-true values and currency rates, and (c) government spending and taxation levels. Analytical solutions are provided in all three paradigms and some useful conclusions are drawn in terms of variable analysis. This short study can be used as an example of how feedback models and stability analysis can be applied as a guideline of 'proofs' in the context of economic policies.

Index Terms — economic policies, public spending, regulated markets, control theory, feedback analysis, minimum wages, currency rates, taxation rates, stable growth.

THERE is a frequently-stated assertion that labor cost is not the driving factor for production cost per unit, even when the selling unit is not a product but some service. However, in times of crisis and austerity, labor costs are almost always the first (and usually the only) factor that is 'relaxed' to lower and lower levels by enterprises, in an effort to keep the margin of profit stable when selling rates decline. Some economists justify these policies as the typical 'rule of thumb': when profits decline, the workers will be paid less and less, until either the business recovers or bankrupts. Others say that its is exactly the recipe of failure, since underpaid workers will rarely work twice as hard to get the business back on it feet - quite the opposite.

Similarly, incentives for new private investments, e.g. low tax rates, are often compared to public spending and the regulatory policies are usually criticized as 'killers' for those incentives. However, there is a definite link between changes in the investment flows and the inherent gap between market value and true value of products and services: excessively positive prospects cause a positive feedback in new investments flow, while the exact opposite happens in times of crises and markets downfall. This oscillatory feedback is (should be) negated by an opposite feedback, usually realized under regulatory policies (e.g. increasing tax rates and government spending), in order to avoid the systemic risk of value 'bubbles' in both the market level and a country's overall GDP change rate.

However, this is not the case in the real world; in fact, the exact opposite happens. This is a 'paradox' of budgeting policies and government spending that are based on false paradigms, hence the end result is typically a self-reinforcing spiral: when things go well, hope and money flow go up too; when things turn bad, more austerity policies lead to the typical 'spiral of death', for a business or a whole country's economy. This is a fundamental issue of high controversy among leading economists and one that will be investigated in-depth under several mathematical formulations in the next sections.

This is by no means a complete paper on economic policies nor in-depth analysis of some of the most important issues in modern economics; it is rather a simple, purely mathematical approach to three very important paradigms, a short study that can be used as an example of how feedback models and stability analysis from classic Control Theory can be applied as a guideline of 'proofs' in the context of economic policies.

The paper is organized as follows: The first section is a short formulation for the problem of minimum wages and their importance in self-regulating markets, under the scope of gain-cost analysis of private firms. Next, the core issues of currency rates and differences between market value and true (hidden) value are investigated under the scope of a first-order feedback model, as well as a stability analysis with regard to private-versus-public spending rates. Finally, government spending, public workers' wages and the general taxation level are described in the context of a differential equations model as mutually dependent variables of the same (economic) system. Analytical solutions are provided in all three paradigms and some useful conclusions are drawn in terms of variable analysis.

Minimum wages and self-regulating labor markets

In order to investigate this 'labor cost shrinkage' dilemma and provide some model-based guidelines, the driving force

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of every profit-focused enterprise can be formulated in correspondence to the 'labor cost' variable. Let $w = \sum p_k$ be the sum of the p_k wages for all k workers, i.e., the enterprise's total labor cost per production unit. Let $NC(w)$ be the total cost per production unit, including labor cost, as a weighted sum of n individual factors. Without loss of generality, assume that the labor cost is at index $i = 1$, hence its weight in the total production cost is α_1 and is singled out from the sum:

$$NC(w) = \sum_{i=1}^{n} \alpha_i \cdot z_i = \alpha_1 \cdot w + \sum_{i=2}^{n} \alpha_i \cdot z_i \tag{1}$$

The net profit $NP(w)$ per sold unit is proportional to the difference between the maximum attainable market (selling) price MPr_{max} and the total cost per unit $NC(w)$, i.e., the maximum margin of profit, while inversely proportional to the total labor cost w:

$$NP(w) \propto \frac{MPr_{max} - NC(w)}{w} \tag{2}$$

Substituting (1) in (2) we have:

$$NP(w) = \frac{MP_{max} - \left(\alpha_1 \cdot w + \sum_{i=2}^{n} \alpha_i \cdot z_i \right)}{w} = \frac{C}{w} - \alpha_1 \tag{3}$$

where $C = MPr_{max} - \sum_{i=2}^{n} \alpha_i \cdot z_i > 0$ is a constant with respect to w and α_i are typical convex weighting factors, i.e.: $\sum_{i=1}^{n} \alpha_i = 1$, $0 \le \alpha_i \le 1$.

In order to find the maximum attainable value for $NP(w)$, the first- and second-order derivatives against w must be calculated:

$$\frac{\partial NP(w)}{\partial w} = \frac{\partial}{\partial w}\left(\frac{C}{w} - \alpha_1\right) = \frac{-C}{w^2} \underset{C>0}{\overset{w \ge w_0 \ge 0}{\Longrightarrow}} \frac{\partial NP(w)}{\partial w} \le 0$$

$$\frac{\partial^2 NP(w)}{\partial w^2} = \frac{\partial}{\partial w}\left(\frac{-C}{w^2}\right) = \frac{2C}{w^3} \underset{C>0}{\overset{w \ge w_0 \ge 0}{\Longrightarrow}} \frac{\partial^2 NP(w)}{\partial w^2} \ge 0$$

In other words, the negative sign of the first-order derivative shows (as expected) that $NP(w)$ decreases as w increases above some minimum allowable value w_0 (employing minimum wages for the workers), while the positive sign of the second-order derivative shows that the function curves upwards. These conditions are typical to all functions $y = x^p$ where $k \le 1$, like in this case. Hence, the function is maximized at the lower limit $w = w_0$:

$$\left.\begin{array}{l} \max_{w} NP(w) \\ s.t.: \ w \ge w_0 \ge 0 \end{array}\right\} \Rightarrow \lim_{w \to w_0} NP(w) = \\ = \lim_{w \to w_0} \left(\frac{C}{w} - \alpha_1\right) = \frac{C}{w_0} - \alpha_1 = NP_{w_0} \tag{4}$$

where NP_{w_0} is the maximum attainable net profit per sold unit with regard to labor cost. Obviously, when there is no minimum-wage limit, i.e., $w_0 = 0$, the net profit w.r.t. labor cost is sky-rocketed to infinity:

$$\left.\begin{array}{l} \max_{w} NP(w) \\ s.t.: \ w \ge w_0 = 0 \end{array}\right\} \Rightarrow \lim_{w \to 0} NP(w) = \\ = \lim_{w \to 0} \left(\frac{C}{w} - \alpha_1\right) = +\infty = NP_0 \tag{5}$$

What (5) proves is nothing new:

> The best labor wages policy, i.e., the one that provides the maximum gain-per-salary ratio, is *slavery* (free labor - no salaries at all).

However, the most important conclusion from this result is the fact that the minimum wage limit, as it is usually legislated by laws and government policies, is actually not something that can be 'discovered' by a totally free (unregulated) market. If salaries can go down to zero, no enterprise has a serious incentive, profit-wise, to offer a decent wage to anyone. In fact, even when minimum wage limits do exist in a labor market, the difference between this lower threshold and the actual mean value of offered wages is only marginal; it only relies on the various enterprises' competitiveness over very few capable candidates for many open positions. Of course, this is hardly the case in the real world where the exact opposite is the rule, i.e., many highly qualified candidates have to compete for a limited number of job openings.

A first-order feedback control model for stable currency rates

The starting point for the following model-based approach is the assertion that there is a distinct difference between the true economy and the currency used in it. That is, each commodity or labor effort has a specific 'hidden' value that is *invariant* with respect to a currency that is used to 'translate' it into monetary value. This assertion is valid for *any* such currency, even those that are bounded to a specific commodity, e.g. gold or silver, since no one commodity can be used a the universal baseline for this evaluation: a huge amount of gold is next to worthless when it can not buy food or water if there are nowhere to be found. On the other hand, a single worker can produce a specific amount of work (on average), just as one apple tree can produce (on the same soil and climate) more or less the same amount of apples, which in turn contain the same amount of nutrients and calories. Hence, the *market price* of a single apple or a single man-hour of work, which is mainly a factor offer and demand, is not a valid invariant metric of its *true value*.

Furthermore, on monetary systems that are based on a representative currency, i.e., some form of printed money or bonds, there is no inherent link between the total volume of currency available to spend and the total volume of commodities and labor to purchase - when there is more money than goods, inflation occurs. More money means higher prices for the exact same commodities, just like less commodities means higher prices at the same total amount of money. It all depends on who controls the total amount of currency available for spending and how this is carefully balanced against the total amount of commodities to purchase.

Basic model

Let V_m be the market price of the commodity, labor effort or even the currency itself and let V_t be the corresponding 'true' value. These two values are correlated linearly by a parameter ρ that corresponds to inflation and deflation effects in the economy:

$$V_m = (\rho + 1) \cdot V_t \tag{6}$$

When $\rho > 0$ then inflation occurs, i.e., the market price is higher than it should for some specific commodity or work effort unit, while $\rho < 0$ means deflation, i.e., the market price of the same commodity gets devalued. Grouping inflation effects as ε^+ and deflation effects as ε^-, the adjustment parameter now becomes $\rho = \varepsilon^+ - \varepsilon^-$.

Let us now consider these two factors, ε^+ and ε^-. During inflation, there is a positive sign in the change rate of the market price V_m against the true value V_t, i.e.:

$$\varepsilon^+ = c^+ \cdot \frac{dV_m}{dV_t} \tag{7}$$

where $c^+ \geq 0$ is a constant. Likewise, during deflation, there is a negative sign in the change rate of the market price V_m against the true value V_t, i.e.:

$$\varepsilon^- = c^- \cdot \frac{dV_m}{dV_t} \tag{8}$$

where $c^- \geq 0$ is a constant. Equations (7) and (8) essentially link the parameter ρ in (6) with the change rate of the market price V_m against the true value V_t. In other words, the inflation and deflation effects are expressed as a function of the first derivative of V_m against V_t.

The terms 'inflation' and 'deflation' here are somewhat misleading, since they are usually linked to increasing and decreasing prices, respectively, in the market. Here, 'inflation' is linked to its primary definition, i.e., the increasing availability and flow of money in the market, a situation that favors 'cheaper' currency, easier loans and incentives for riskier investments. This means that when the money flow increases, *into* the economy (private investments, low taxation levels, economy growth, etc), the incentives for moving even more money into it increases, since profitable businesses are plenty. On the other hand, 'deflation' here is also linked to its primary definition, i.e., the decreasing availability and flow of money in the market, a situation that results in 'expensive' currency, harder loans and incentives for more conservative (or no) investments. This means that when the money flow decreases, *out of* the economy (outgoing foreign exchange, high taxation levels, economy shrinking, etc), the incentives for putting more and more money away from it increases, since profitable investments are scarce. In other words, equations (7) and (8) translate the momentum of V_m against V_t into quantifiable feedback, reinforcing (positive) or dampening (negative), according to the relation between c^+ and c^-.

There is also a definite link between inflation/deflation and interest rates in bank loans: during inflation, the governments and central banks try to 'slow down' excessive loaning and credit card use by raising the baseline for interest rates, while during deflation they try to 'boost' the economy by lowering this baseline and thus enabling more money flow in the market. Here, these 'correcting' actions are essentially included in both c^+ and c^-, according to the direction these authorities want to employ into (7) and (8) and, in the end, into (6) as well.

Analytical solution

The model described by (6) now becomes a differential equation that is to be solved, i.e., fully defines V_m as a function of V_t. The following steps show how:

$$\begin{aligned}
V_m &= (\rho + 1) \cdot V_t = (\varepsilon^+ - \varepsilon^- + 1) \cdot V_t \\
&= \left((c^+ - c^-) \cdot \tfrac{dV_m}{dV_t} + 1\right) \cdot V_t \\
\Leftrightarrow \quad & (c^+ - c^-) \cdot V_t \cdot \tfrac{dV_m}{dV_t} - V_m + V_t = 0 \\
\Leftrightarrow \quad & (c^+ - c^-) \cdot x \cdot \tfrac{dy}{dx} - y + x = 0 \\
\Leftrightarrow \quad & \tfrac{dy}{dx} + \left(\tfrac{-1}{c^+ - c^-}\right) \cdot \tfrac{y}{x} + \left(\tfrac{1}{c^+ - c^-}\right) = 0
\end{aligned}$$

$$\Leftrightarrow \quad \frac{dy}{dx} = \beta \cdot \frac{y}{x} - \beta = F\left(\frac{y}{x}\right) \tag{9}$$

where $y = V_m$, $x = V_t$ and $\beta = \frac{1}{c^+ - c^-}$. The last step in the previous sequence is essentially a typical transformation into a well-known form of differential equations that can be solved by substituting $z = \frac{y}{x}$ and calculating the integral:

$$\begin{aligned}
& \tfrac{dy}{dx} = \beta \cdot \tfrac{y}{x} - \beta = F\left(\tfrac{y}{x}\right) \\
\left(z = \tfrac{y}{x}\right) \quad & \Rightarrow \ln x = \int \tfrac{dz}{F(z) - z} + \alpha \\
& \ln x = \int \tfrac{dz}{F(z) - z} = \int \tfrac{d\left(\frac{y}{x}\right)}{\left(\beta \cdot \frac{y}{x} - \beta\right) - \left(\frac{y}{x}\right)} \\
& = \int \tfrac{d\left(\frac{y}{x}\right)}{\frac{y}{x} \cdot (\beta - 1) + (-\beta)} = \int \tfrac{dz}{z \cdot (\beta - 1) + (-\beta)} \\
& = \tfrac{1}{(\beta - 1)} \cdot \ln\left(z \cdot (\beta - 1) + (-\beta)\right) \\
\Leftrightarrow \quad & \ln x = \tfrac{1}{(\beta - 1)} \cdot \ln\left(z \cdot (\beta - 1) + (-\beta)\right) \\
\Leftrightarrow \quad & \ln x^{(\beta - 1)} = \ln\left(\tfrac{y}{x} \cdot (\beta - 1) + (-\beta)\right) \\
\Leftrightarrow \quad & x^{(\beta - 1)} = \tfrac{y}{x} \cdot (\beta - 1) + (-\beta) \\
\Leftrightarrow \quad & (\beta - 1) \cdot y = x^\beta + \beta \cdot x
\end{aligned}$$

$$\Leftrightarrow \quad y = \left(\frac{1}{\beta - 1}\right) \cdot x^\beta + \left(\frac{\beta}{\beta - 1}\right) \cdot x \tag{10}$$

Hence, from the final result an analytical formula of V_m with regard to V_t becomes available:

$$V_m = \left(\frac{1}{\beta - 1}\right) \cdot V_t^\beta + \left(\frac{\beta}{\beta - 1}\right) \cdot V_t \qquad \beta = \frac{1}{c^+ - c^-} \tag{11}$$

Note that (11) is more or less the long-term expansion of (6). That is, (6) is the 'instance' definition of V_m with regard to V_t as a differential equation, while in (11) V_m is defined only as a function of V_t (no differentials) and some constant parameters. This analytical form is appropriate for calculating stability and feedback factors as the result of these constant parameters and how they affect the relation between V_m and V_t.

As a verification step, one can calculate the differential term $\frac{dy}{dx}$ by its starting definition in (9) and by its analytical solution in (10). From (9) this calculation gives:

$$\frac{dy}{dx} = \beta \cdot \frac{y}{x} - \beta = \beta \cdot \left(\frac{x^{\beta}+\beta \cdot x}{\beta-1}\right) \cdot x^{-1} = \dots = \left(\frac{\beta}{\beta-1}\right) \cdot \left(x^{\beta-1}+\beta\right)$$

Similarly, from (10) the same calculation gives:

$$\frac{dy}{dx} = \left(\frac{1}{\beta-1}\right) \cdot \frac{d\left(x^{\beta}+\beta \cdot x\right)}{dx} = \frac{\beta \cdot x^{\beta-1}+\beta}{\beta-1} = \left(\frac{\beta}{\beta-1}\right) \cdot \left(x^{\beta-1}+\beta\right)$$

Hence, (11) is a valid analytical solution of (6).

Stability and feedback analysis

Equation (11) provides a full description for V_m with regard to V_t and the means to analyze its asymptotic behavior. Since β is the only parameter that includes all model configuration, it is the main factor of interest here. Specifically, it is evident that as $|\beta|$ increases, i.e., as the difference $|c^+ - c^-|$ becomes smaller, V_m exhibits larger exponent in V_t. This means that V_m either increases at higher rates (when $\beta > 0$) or drives the first term to zero (when $\beta < 0$). Furthermore, as $|\beta|$ becomes larger, the coefficient $\frac{\beta}{\beta-1}$ of the second term in (11) approaches unity. In other words, as c^+ approaches c^-, V_m and V_t become directly proportional (not just linearly dependent).

Combining these previous comments with respect to β, it is clear that if V_m is to be 'stabilized' against V_t, β can be selected accordingly in order to diminish the first (higher-order) term and reinforce the second (linear) term. This happens only when $|\beta| \gg 1$ and $\beta < 0$, i.e., as $\beta \to -\infty$:

$$V_m = \left(\frac{1}{\beta-1}\right) \cdot V_t^{\beta} + \left(\frac{\beta}{\beta-1}\right) \cdot V_t \overset{\beta \to -\infty}{\longrightarrow} V_t^{+} \qquad (12)$$

where V_t^+ means that it is approached from higher values as β becomes more and more negative. In other words:

$$\beta \to -\infty \Leftrightarrow c^+ - c^- \to 0^- \Longrightarrow V_m \to V_t^+ \qquad (13)$$

What equation (13) says is pretty clear:

> If market prices of commodities and work effort are to be kept close to their true values, the negative feedback factors (high taxes, high interest rates, high government spending, etc) should *closely match* the positive feedback factors (low taxes, low interest rates, strict government spending, etc).

This result is not something unexpected; keeping taxes and interest rates high is the standard policy for slowing down a booming economy into a 'controlled growth' state. This is necessary to avoid excessive debt increase in both public and private sector, as well as decreasing the incentives of 'bubbles' in stock markets. However, what is very interesting is that the proper control policy is for the authorities to *counter match* the positive feedback with *proportional* negative feedback actions. In other words, government spending, taxation levels and interest rates should *always* increase/decrease in proportion to private investments, creation of new businesses and incoming flow of foreign capital.

Unfortunately, the idea of a deliberate slowdown in the economy is something that is often unthinkable for modern free trades and stock markets - this is why (12) and (13) also constitute a very realistic explanation of the various financial crises, like the dot-com bubble of the late '90s or the 2008 house market crash in USA: *when negative feedback is not enforced, the catastrophic deviation of V_m from V_t becomes a mathematical certainty*.

Government spending and stable taxation level

One of the most controversial issues in all economic models is the acceptable amount of government spending for public services, infrastructure and government salaries. In general, the amount of government budget available for spending is directly proportional to the country's Gross Domestic Product (GDP):

$$\{GDP\} = C + I + G + (X - M) \sim S + P = W \qquad (14)$$

Basic model

Equation (14) is the typical calculation of GDP: C is for consumption, I is for investments and savings (domestic), G is for government spending and $(X - M)$ is the exports-imports (net) balance. Reformulating these parameters, let W be the GDP portion attributed to salaries of workers in the public and the private sector, i.e., S and P respectively. Public spending on infrastructure can be reformulated as the weighted sum of N factors, each contributing α_i to the total spending:

$$G_s = \sum \alpha_i g_i \quad i = 1, \dots, N$$

$$\sum \alpha_i = 1 \quad 0 \le \alpha_i \le 1$$

Similarly, a 'public welfare' index can be calculated as a weighted sum of K factors, each contributing γ_i to the index, that affect availability of public services to the people:

$$G_w = \sum \gamma_k (1 - h_k) \quad k = 1, \dots, K$$

$$\sum \gamma_k = 1 \quad 0 \le \gamma_k \le 1$$

where h_k is the fraction of people without access to public service k. Clearly, there is a link between G_s and G_w, i.e., $G_w \sim G_s$.

Let us now focus on the government budget that need to cover for public workers' salaries S and public spending G_s. Let Q^+be the positive flow, which is essentially the sum of taxes on wages to all workers (public and private sector), and let Q^- be the negative flow, which goes to S and public spending G_s. If $0 \leq c \leq 1$ is the balancing factor between salaries and infrastructure in government spending and p is the balancing factor between private sector and public sector fractions in the total work force, then:

$$Q^+ = (S + P) \cdot t = (W \cdot (1-p) + W \cdot p) \cdot t = W \cdot t$$

$$Q^- = (1-c) \cdot \hat{S} + G_s \cdot c = (1-c) \cdot (1-t) \cdot S + G_s \cdot c$$

For a long-term viable budget management without deficiencies, huge reserving and external loans, then the positive/negative flows should be roughly equal:

$$Q^+ \simeq Q^- \Leftrightarrow S \cdot t + P \cdot t \simeq (1-c) \cdot (1-t) \cdot S + G_s \cdot c$$

$$Q^+ - Q^- = W \cdot t - (1-c) \cdot (1-t) \cdot (1-p) \cdot W - G_s \cdot c = \varphi \quad (15)$$

where φ is the instantaneous (annual) balance in the government budget. The model presented above assumes perfect mechanisms for spending, paying salaries and collecting taxes. For a more realistic calculation, deficiency factors have to be introduced in all the major components in (15), i.e.:

$$\hat{t} = t \cdot (1-\varepsilon_t) \ , \ 0 \leq \varepsilon_t \leq 1 \ , \ 0 \leq t \leq 1$$

$$\hat{W} = W \cdot (1-\varepsilon_w) \ , \ 0 \leq \varepsilon_w \leq 1$$

$$\hat{G} = G \cdot \left(1-\varepsilon_g\right) \ , \ 0 \leq \varepsilon_g \leq 1$$

$$\hat{\varphi} = \varphi \cdot \left(1-\varepsilon_\varphi\right) \ , \ 0 \leq \varepsilon_\varphi \leq 1$$

Here, ε_t stands for deficiency in collecting taxes, ε_w stands for deficiency in work effort (outsourced workers), ε_g stands for deficiency in constructing and maintaining public services (corruption) and ε_φ stands for deficiency due to inflation (domestic currency devaluation). These adjusted components can be introduced directly into (15) for proper calculations; however, for the shake of simplicity, the simplified model of (15) will be used as-is, since this choice does not affect the analysis that follows.

Analytical solution

Returning now to (14), the investments component I can be expressed as a factor of W, meaning that the amount of money available for domestic spending drives the incentives for more investments, new businesses and foreign capitals:

$$I = \xi \cdot W \cdot (1-t) \cdot (1+\vartheta) \quad \vartheta, \xi \geq 0 \qquad (16)$$

where ξ is the amount of available money $W \cdot (1-t)$ (after taxation) that goes into investments and ϑ is the multiplier that is attributed to foreign capital that comes into the domestic economy as investments too. Hence, the true annual change in W can now be stated as a function of φ and I as:

$$\begin{array}{rcc} \triangle W & = & \varphi + I \\ & = & W \cdot t - (1-c) \cdot (1-t) \cdot (1-p) \cdot W - G_s \cdot c \\ & + & \xi \cdot W \cdot (1-t) \cdot (1+\vartheta) \end{array} \qquad (17)$$

Equation (17) is a differential model that links W with its change rate and all the other factors. Since it is stated in a discrete form (annual changes), it can be solved as a first-order iterative equation, defining $a_n = W_n$ and substituting for all the other factors:

$$a_{n+1} = a_n \cdot (A+B) + C \Leftrightarrow a_{n+1} - a_n \cdot (A+B) = C \qquad (18)$$

$$A = t - (1-c) \cdot (1-t) \cdot (1-p) \qquad (19)$$

$$B = \xi \cdot (1-t) \cdot (1+\vartheta) \qquad (20)$$

$$C = -G_s \cdot c \qquad (21)$$

Equation (18) is solved by calculating the solution of the corresponding homogeneous system ($C = 0$) and then trying a solution similar to the right-hand side of the general equation. The solution of the homogeneous system is:

$$d_{n+1} - d_n \cdot (A+B) = 0$$

$$\lambda - (A+B) = 0 \Rightarrow \lambda = A + B \Rightarrow d_n = (A+B)^n \cdot d_0 \qquad (22)$$

Since the right-hand side of (18) is a zero-order polynomial, a constant can be introduced as a solution to the general equation:

$$\hat{a} = b_0 \Rightarrow b_0 - (A+B) \cdot b_0 = C \Leftrightarrow b_0 = \frac{C}{1-(A+B)} \qquad (23)$$

Then, the complete solution of (18) is the sum of the partial solutions of (22) and (23), i.e.:

$$a_n = d_n + \hat{a} = (A+B)^n \cdot d_0 + b_0 \qquad (24)$$

where d_0 is a constant that can be calculated directly by using any value for n, i.e.:

$$n = 0 \rightarrow a_0 = 1 \cdot d_0 + b_0 \Leftrightarrow d_0 = a_0 - b_0 \quad (25)$$

Substituting (23) and (25) into (24), the final solution for a_n becomes:

$$\begin{aligned} a_n &= (A+B)^n \cdot (a_0 - b_0) + b_0 \\ &= (A+B)^n \cdot \left(a_0 - \tfrac{C}{1-(A+B)}\right) + \tfrac{C}{1-(A+B)} \end{aligned}$$

or in terms of the original W parameter:

$$W_{n+1} = (A+B)^n \cdot \left(W_0 - \frac{C}{1-(A+B)}\right) + \frac{C}{1-(A+B)} \quad (26)$$

where W_0 is a constant corresponding to some starting value for W. Hence, the total amount of money available as workers' salaries (public and private sectors) is now expressed as a function of all the other parameters of (15) and (16).

Stability and feedback analysis

In order to evaluate the stability constraints for the model described in (26), the most important factor is the base of the exponent, i.e., $(A+B)$. The same result can be drawn by applying the z-transformation to the original model in (18):

$$W_{n+1} - (A+B) \cdot W_n = C \xrightarrow{F(z)} H(z)$$

$$H(z) = \frac{C}{1-(A+B)\cdot z^{-1}} \longleftrightarrow h(n) = C \cdot (A+B)^n \cdot u(n) \quad (27)$$

It is clear from (27) that, in order for the system to be stable, the constraint $|A+B| \leq 1$ needs to be true in all cases. Let us now examine the case $A+B \leq 1$ with regard to the taxation level t, applying (19) for A and (20) for B:

$$\begin{aligned} & A+B & \leq 1 \\ \Leftrightarrow\ & t-(1-c)\cdot(1-t)\cdot(1-p)+\xi\cdot(1-t)\cdot(1+\vartheta) & \leq 1 \\ \Leftrightarrow\ & t\cdot(1+(1-c)\cdot(1-p)-\xi\cdot(1+\vartheta)) & \\ & -((1-c)\cdot(1-p)-\xi\cdot(1+\vartheta)) & \leq 1 \\ \Leftrightarrow\ & t\cdot(1+\tau)-\tau & \leq 1 \end{aligned}$$

where:

$$\tau = (1-c)\cdot(1-p)-\xi\cdot(1+\vartheta) \quad (28)$$

and finally we get:

$$t \leq \frac{1+\tau}{1+\tau} = 1 \quad (29)$$

Similarly, for the lower bound we get:

$$\begin{aligned} & A+B & \geq -1 \\ \Leftrightarrow\ & t-(1-c)\cdot(1-t)\cdot(1-p)+\xi\cdot(1-t)\cdot(1+\vartheta) & \geq -1 \\ \Leftrightarrow\ & t\cdot(1+\tau)-\tau & \geq -1 \end{aligned}$$

which gives:

$$t\cdot(1+\tau)-\tau \geq -1 \Leftrightarrow t \geq \frac{-1+\tau}{1+\tau} \quad (30)$$

Combining (29) and (30), and since $0 \leq t \leq 1$, we get the final range for 'stable' taxation level:

$$\max\left\{0, \frac{-1+\tau}{1+\tau}\right\} \leq t \leq 1 \quad (31)$$

Equation (31) is essentially a range constraint for t and describes the stability conditions for W with respect to the taxation level. In practice, if the lower bound becomes positive, this range is shrinking towards the upper bound, i.e., the taxation levels are forced to be higher in order to maintain a stable budget management. Using (28) this translates to:

$$\frac{-1+\tau}{1+\tau} > 0 \Leftrightarrow \tau > 1 \Leftrightarrow (1-c)\cdot(1-p) > 1+\xi\cdot(1+\vartheta) \quad (32)$$

The result stated by (32) is indeed a very interesting one:

> For a stable spending of government budget, the available range for the taxation level is *shrinking towards the higher limit* as the total number of public workers and/or the spending weight (i.e., wages level) of their salaries becomes larger than the domestic and foreign investments (incoming flow).

This statement per-se is completely expected, as government budget comes from taxes and foreign capital investments (assuming no long-term policies for debt deficiencies are allowed). However, if the total government spending is assumed constant, (32) states that the system can also be stabilized by allocating more funds to public infrastructure and services instead of public workers' wages. In other words, *budgets cuts (austerity) is not necessarily the only solution available.*

Equation (32) incorporates c and p as negative terms, while ξ and ϑ as positive ones, hence it is fairly easy to come with another interesting result: Since p corresponds to the fraction of work force employed in the private sector, (32) implies that having an excessively large number of private sector workers, with regard to true investments, essentially destabilizes the system. This assertion can be explained by the fact that excessive private worker force means excessive sum of salaries available for spending, thus increased attraction of domestic and foreign funds for new investments. This incentive essentially destabilizes the control of $\triangle W$ in (17) and may cause a catastrophic oscillation (market bubbles).

Therefore, *the proper stabilization action is for the government to 'slow down' any excessive increases*, a result similar to the one stated at the end of the previous paradigm (see previous section). This action is usually executed by employing higher taxes and limiting incoming flow of investment funds - policies that are usually considered unthinkable for modern free trades and stock markets.

Conclusion

In this paper, simple mathematical models from Control Theory were applied to three very important economic paradigms, namely (a) minimum wages in self-regulating markets, (b) market-versus-true values and currency rates, and (c) government spending and taxation levels.

The main conclusions are:

- The best labor wages policy, i.e., the one that provides the maximum gain-per-salary ratio, is *slavery* (free labor - no salaries at all).
- Even when minimum wage limits do exist in a labor market, the difference between this lower threshold and the actual mean value of offered wages is only marginal.
- If market prices of commodities and work effort are to be kept close to their true values, the negative feedback factors (high taxes, high interest rates, high government spending, etc) should *closely match* the positive feedback factors (low taxes, low interest rates, strict government spending, etc).
- For stable economies, the proper control policy is for the authorities (government) to *counter match* the positive feedback (private investments) with *proportional* negative feedback actions.
- For a stable spending of government budget, the available range for the taxation level is *shrinking towards the higher limit* as the total number of public workers and/or the spending weight (i.e., wages level) of their salaries becomes larger than the domestic and foreign investments (incoming flow).
- If the total government spending is assumed constant, the budget can also be stabilized by allocating more funds to public infrastructure and services instead of public workers' wages (i.e. wages cuts is not the only 'correcting' solution).

This short study that can be used as an example of how feedback models and stability analysis can be applied as a guideline of 'proofs' in the context of economic policies.